Displacements on the regular tessellations of the two- or three-dimensional space

Host Laboratories

- Laboratoire d’Informatique Gaspard-Monge (LIGM - UMR 8049), Université Paris-Est
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Main project members

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Practical Information

- **Funding:** Labex Bézout, Université Paris-Est
- **Salary:** about 2000 euros/month net (varied depending on the level of expertise and experience)
- **Starting date:** fall 2017
- **Duration:** 2 years
- **Application:** send the following documents to yukiko.kenmochi@u-pem.fr and pascal.romon@u-pem.fr:
  1. CV,
  2. List of publications,
  3. Summary of research achievements,
  4. Future plan of research with possible links to the current project
  5. Contact details of at least two references who can provide recommendation letters upon request.

Project Description

Context

An image (a scene or an object) is mathematically represented by a function defined on a domain of $\mathbb{R}^n$, $n = 2, 3$. In order to store, process and manipulate such an image by computer, it is digitized by partitioning the support $\mathbb{R}^n$ into regular polygons or polyhedra: for example, the square, hexagonal or triangular tessellation of $\mathbb{R}^2$ and the cubical tessellation of $\mathbb{R}^3$ (see Figure 1). In particular, the center of each cell of the square or cubical tessellation is represented by a point of $\mathbb{Z}^n$, so that the support of a digital image is $\mathbb{Z}^n$ instead of $\mathbb{R}^n[1]$. This operation is called *digitization*, defined as a rounding function, $D : \mathbb{R}^n \rightarrow \mathbb{Z}^n$.

![Figure 1: Regular tiling of the two-dimensional space: (a) the square tiling, (b) the hexagonal tiling, and (c) the triangular tiling.](image-url)
Figure 2: Digital images (left) and their displacements (right). (a) A disk preserves neither topology nor geometry, while (b) a half-plane preserves its topology but not its geometry. (c) A thin digital plane preserves neither topology nor geometry, while (d) a thicker digital plane preserves topology (connected without hole) but is not plane. (e) A binary retinal image here preserves neither its topology nor its geometry.

Displacements, i.e. orientation-preserving isometries, obtained by composing translations and rotations, which constitute a subgroup of the isometry group Isom(n), are among the most fundamental transformations in digital image processing. They are often involved in image registration, motion tracking, and shape recognition in two- and three-dimensional digital images. In order to apply a displacement $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ to digital images, it must be digitized as $T := D \circ \mathcal{T}|_{\mathbb{Z}^n}$. However, while the geometry and topology are preserved by any displacement $\mathcal{T}$ in $\mathbb{R}^n$, in the discrete domain, i.e. $\mathbb{Z}^n$ but not $\mathbb{R}^n$, these geometric and topological invariances are generally lost because of the discontinuities induced by the digitization processes of the function $T : \mathbb{Z}^n \to \mathbb{Z}^n$ (see Figure 2). Indeed, these defects are caused by the loss of the bijectivity of $T$ (see Figure 3).

Objectives and challenges

This problem has not been studied in a systematic and generic way until now. It is, however, of crucial importance, particularly when images contain rich contents such as a large number of objects structured in a complex way, for example in biomedical and geomaterial applications.

At present, this issue is addressed through ad hoc strategies in most cases. Thus, displacements are generally applied in $\mathbb{R}^n$, after the digital images have been plunged from $\mathbb{Z}^n$ into this continuous space. The results are then re-digitized in $\mathbb{Z}^n$. This succession of spatial transpositions induces numerical approximations which necessarily lead to inaccurate results.

In order to avoid losing such control of geometry and topology in $\mathbb{Z}^n$, we aim to develop a purely discrete framework for displacements on $\mathbb{Z}^n$ based on digital geometry and topology concepts. This new framework will allow to manipulate discrete objects in a rigid manner via “pixel-by-pixel” operations.
Sub-themes and approaches

It should be noted that there are approaches based on approximating bijections of $\mathbb{R}^n$ by bijections of $\mathbb{Z}^n$. It can be shown that some behave well (for example, approximating rotations by quasi-shear mappings $[^2]$). In this project, however, we consider the model of the exact digitized displacements $T$ and seek solutions to the following sub-problems.

(1) Topological problem

A subset of this problem has been studied in $\mathbb{Z}^2$. The class of two-dimensional images that preserve their topological properties during displacements – called regular images – has been identified, as well as methods allowing such “regularization” $[^3]$. Nevertheless, these strategies in $\mathbb{Z}^2$ rely on topological models that are not intrinsically transposable to higher dimensions $[^4]$. Consequently, it will be essential to develop new strategies more generic and thus valid in any dimension.

(2) Local analysis of displacements

A combinatorial model of the local behavior of displacements in $\mathbb{Z}^2$ allows us to study the topological preservation and bijectivity of displacements in $\mathbb{Z}^2$ $[^5, 6, 7]$. In order to construct a combinatorial model of the local behavior of displacements on $\mathbb{Z}^n$, the transformations can be classified according to their effect on a digital image patch. This classification can be interpreted as specific problems of hypersurface arrangements in the parameter space of the transformations, some of which can be solved by methods of computational geometry. For the calculation, despite the existence of degenerate cases, the algorithm based on plane sweeping is proposed for $n = 2$, and it can be applied to an image patch whose cardinality is greater than 100 $[^8]$. On the other hand, working in $\mathbb{Z}^3$ increases considerably the complexity of the problem, and makes it difficult to apply directly classical techniques such as the cylindrical algebraic decomposition or the critical point method. However, this problem can be reduced to the calculation of sampling points in an arrangement of quadrics in the parameter space of rotations; then, information on the other parameters of translations is retrieved $[^9]$. An ad hoc algorithm based on this method has been implemented for an image patch of cardinality 7, but new algorithms that are more efficient will need to be developed for larger patches.

(3) Characterization/certification of the bijectivity

Although the displacements are bijective and isometric in $\mathbb{R}^n$, these properties are lost during the digitization in $\mathbb{Z}^n$ (see Figure 2). To study these defects, we extend a combinatorial model of the local behavior of displacements in $\mathbb{Z}^2$. This allows us to study the bijective displacements in $\mathbb{Z}^2$, and to propose algorithms for verifying if a given displacement is bijective restricted to a fixed finite subset of $\mathbb{Z}^2$ $[^7]$.

In the case of displacements in $\mathbb{Z}^3$, more precisely, of discrete rotations, a similar characterization does not exist. On the other hand, an algorithm to certify the (global) bijectivity of digitized rational rotations was obtained using the arithmetic properties of Lipschitz quaternions $[^10]$. This algorithm also makes it possible to show the existence of bijective digitized rotations whose axis of rotation does not correspond to one of the coordinate axes. Nevertheless, the semi-dense or discrete characteristics of the image of a lattice by a digitized rational rotation is still an open question.

This arithmetic approach is promising because it can also handle other two-dimensional regular tilings $[^11]$. 

(4) Geometric problem

In addition, it will be pertinent to explore also questions of discrete differential geometry, such as the behavior of discrete curvature under displacements. In this context, new notions of "geometric" and "topological" preser-
vation must be proposed. For example, the approach of "integral geometry" [12], which allows us to define a measure of the continuous shapes digitized in the same digital shape, can be a good mathematical tool in order to search for displacements in $\mathbb{Z}^n$ preserving this type of measure.

(5) **Link to the quasi-affine transformations**

There are generic approaches based on the quasi-affine transformations [13, 14], which show that the digitization of a continuous operation is sometimes annoying (loss of bijection), but sometimes very rich with the associated arithmetic structure (tilings, enumeration, etc.). It will be naturally interesting to link these approaches to the study of displacements in $\mathbb{Z}^n$.

**References**


